Correlation functions of twist operators applied to single self-avoiding loops

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Abstract

The O(n) spin model in two dimensions may equivalently be formulated as a loop model, and then mapped to a height model which is conjectured to flow under the renormalization group to a conformal field theory (CFT). At the critical point, the order n terms in the partition function and correlation functions describe single self-avoiding loops. We investigate the ensemble of these self-avoiding loops using twist operators, which count loops which wind non-trivially around them with a factor -1. These turn out to have level two null states and hence their correlators satisfy a set of partial differential equations. We show that partly-connected parts of the four point function count the expected number of loops which separate one pair of points from the other pair, and find an explicit expression for this. We argue that the differential equation satisfied by these expectation values should have an interpretation in terms of a stochastic(Schramm)-Loewner evolution (SLE$_\kappa$) process with $\kappa = 6$. The two point function in a simply connected domain satisfies a closely related set of equations. We solve these and hence calculate the expected number of single loops which separate both points from the boundary.

1 Introduction

In a recent paper, Werner [1] has shown there exists a measure on simple loops on any Riemann surface which has the property of conformal restriction. This is to say that if $D' \subset D$ are any two subdomains of the manifold, the measures on loops in D' obtained by (a) restriction of the measure on loops in D to those in D', and (b) conformally mapping $D \to D'$, are the same. Moreover this measure is unique up to multiplication by a constant. We refer to these loops throughout the paper as self-avoiding loops and to the mass of any subset under Werner's measure as the μ-mass.

One may also consider the set of self-avoiding polygons on some regular lattice embedded in the manifold. The total number of such polygons of length l is known to grow as μ^l where μ is lattice dependent. The measure which weights each polygon with a factor x_c^l (where $x_c = \mu^{-1}$) has the restriction property,

and is commonly conjectured also to be conformally invariant in the limit of vanishing lattice spacing, which means that it should give a particular example of Werner's measure. In this paper we assume this to be true, and hence conjecture values of the μ-mass of certain loop subsets, using (non-rigorous) Coulomb gas and CFT methods applied to a generalised model, the $\mathrm{O}(n)$ model.

The $\mathrm{O}(n)$ model encompasses a large group of physically interesting models, including the Ising spin model, the percolation problem and self-avoiding loops. The theory may be written as a spin model with nearest neighbour interactions, or alternatively in the loop gas picture, in which the states of the model are non-intersecting loops constructed on the edges of the spin model. Each loop is weighted with a factor nx^l, where x is related to the reduced coupling in the model and l is the length of the loop. In the Ising model, these loops are the cluster boundaries of a spin model defined on the dual lattice. At the critical point of the model, $x = x_c(n)$, it is conjectured to flow under the renormalization group to a Gaussian free field [2, 3], which is a well known example of a conformal field theory (CFT). One of us [4] showed the existence of operators in this theory whose two point correlation functions count loops around one of two points with a weight n' rather than n, hence giving information about the distribution of loops. The choice $n' = -n$ is of particular interest; these operators are then called twist operators and are at $(r, s) = (1, 2)$ in the Kac classification [5]. Hence, they have null states at level two. The two point correlation function is then equivalent to the expectation value of $(-1)^N$ in the loop ensemble, where N is the number of intersections of loops with a defect line, a continuous curve connecting the two points. The twist operators may therefore be thought of as a source and a sink for this defect line.

Self-avoiding loops are described by the loop gas picture of the $\mathrm{O}(n)$ model with $n \to 0$. The partition function and correlation functions are determined (to order n^1) by graphs with just a single loop. The two point correlation function of twist operators therefore leads to an analytic expression for the number of single loops which separate the locations of the operators, weighted by x_c^l as in figure 1. This number is logarithmically divergent in the continuum limit, however, due to the contribution from vanishingly small loops around each of the points.

In this paper, we consider the four point correlation function of these twist operators on the Riemann sphere. In the loop gas picture, we may think of each operator as again being a source (or sink) of a defect line. Hence there is a defect line running between each pair of points. We argue that the choice of paths for the defect lines is unimportant and that they may run between any two distinct pairs of points. The four point correlation function can then be shown to be the expectation value of $(-1)^N$ where N is now the total number of crossings of loops with defect lines. Particular semi-connected parts of this four point function yield the expected numbers of weighted loops (the μ-mass of loops) which wind around two of the four locations of the operators, as in figure 2. CFT may be used to derive the form of the four point function, since the null states of the operators imply that the correlation functions obey a set of Belavin, Polyakov and Zamalodchikov (BPZ) type partial differential equations [6]. We solve these equations and hence find analytic expressions for the μ-mass of loops which separate one pair of points from the other. For loops

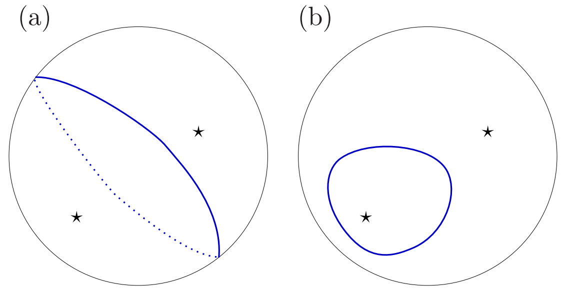

Figure 1: Two examples of single loops winding around one of two operators for the theory on the Riemann sphere. The stars mark the locations of the operators and the dashed line indicates that the curve wraps around the back of the sphere. Two point function is the sum of such loops, weighted by x_c to the power of their length. Notice that the sum contains loops with vanishingly small perimeter in the continuum limit.

which separate (z_1, z_2) from (z_3, z_4), for example, we obtain the expression

$$\frac{-1}{24\pi}\Big(\eta\,{}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta)+\overline{\eta}\,{}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\overline{\eta})\Big) \\ +\frac{2^{1/3}\pi}{3\sqrt{3}\Gamma(\frac{1}{6})^2\Gamma(\frac{4}{3})^2}|\eta(1-\eta)|^{2/3}|\,{}_2F_1(\frac{2}{3},1,\frac{4}{3},\eta)|^2\,,$$

where $\eta \equiv (z_1-z_2)(z_3-z_4)(z_1-z_3)^{-1}(z_2-z_4)^{-1}$ is the cross-ratio of the points.

These expected numbers of weighted loops are finite in the continuum limit because there is no contribution from vanishingly small loops, and are invariant under conformal transformations. We show that the non-leading behaviour of these expressions as $\eta \to 0$ reveals the derivative of the central charge with respect to n at $n = 0$. This is consistent with the interpretation [7] of the stress tensor as the spin-two component of the relative probability that a loop passes between two points in the limit $z_{12} \to 0$.

We use similar arguments for the O(n) model in a simply connected domain. Such a domain may always be mapped via a conformal transformation to the upper half plane with the real axis as the boundary. The two point function in this domain satisfies the same partial differential equations as the four point function in the bulk, with the locations of the two operators and the reflections of these points in the real axis being the locations of the four operators in the bulk theory. We show that the connected two point function of twist operators in the $n \to 0$ limit counts the mass of loops around the locations of both operators, and find the explicit expression

$$-\frac{1}{12\pi}\ln(\eta(1-\eta))-\frac{1}{12\pi}\eta\,{}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta) \\ +\frac{\Gamma(2/3)^2}{6\pi\Gamma(4/3)}(-\eta(1-\eta))^{\frac{1}{3}}\,{}_2F_1(\frac{2}{3},1;\frac{4}{3};\eta)\,,$$

where η is now the cross ratio of the two points together with their conformal images in the boundary.

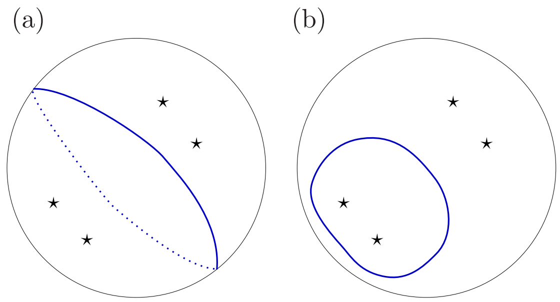

Figure 2: Two examples of single loops winding around the locations of two of four operators for the theory on the Riemann sphere. A particular semi-connected four point correlation function counts the number of such loops, weighted by x_c to the power of their length. Note that, in contrast to the two point correlation function above, there are no vanishingly small loops contributing to the sum.

The layout of this paper is as follows. In the next section, we recall the arguments leading to the Coulomb gas picture of the O(n) model. In section 3 we discuss twist operators in the O(n) model and derive the form of the two point correlation function of these operators. The small n limit of the theory is explored and the correlation function in this limit is shown to depend (to order n^1) only on the configurations of a single loop. In section 4 we discuss the four point correlation function of twist operators. Because the twist operators have null states at level two, this correlation function satisfies a set of partial differential equations, which we solve analytically. The interpretation of the four point function in terms of a pair of defect lines is explained, followed by the example of the four point function of twist operators as the four point function of spin operators in the Ising model. Section 4.1 is devoted to the small n limit of the four point function and contains a derivation of the mass of loops winding around two of the four points. We then explain how these numbers, as functions of the positions of the four points, hold a key to measuring the effective central charge of the O(n) model as $n \to 0$. In section 6 we apply the theory of twist operators to the O(n) model in a simply connected domain and show that the mass of loops around two points in such a domain is invariant under conformal transformations and finite in the limit of vanishing lattice spacing. Finally, in section 7 we show that the mass of loops around two of four points on the Riemann sphere can be thought of in terms of an SLE$_6$ process.

2 The Coulomb Gas

Let us start with the following partition function for the O(n) model [2]:

$$Z_{O(n)} = \mathrm{Tr} \prod_{(ij)} \left(1 + x\mathbf{s}(r_i) \cdot \mathbf{s}(r_j)\right). \tag{1}$$

The $\mathbf{s}(r_i)$ are n-component spins on a lattice $\{r_i\}$ and the product is over pairs of nearest neighbours. The product in the partition function can be expanded into a sum of 2^N terms, where N is the number of nearest neighbours. Each term can be represented by a graph of open and closed loops in the following way: the neighbouring sites, i and j, are joined together by a line if the $x\mathbf{s}(r_i)\cdot\mathbf{s}(r_j)$ term was chosen in the expansion of equation (1), or left disconnected if the number 1 was chosen instead. The trace over spins may then be done for each graph individually. Since $\mathrm{Tr}\,\mathbf{s}(r_i)^{\mathrm{odd}}=0$ because of the symmetry under the transformation $\mathbf{s}(r_i)\to-\mathbf{s}(r_i)$, all graphs containing an odd number of $\mathbf{s}(r_i)$ make no contribution to the partition function. These are the graphs with open loops, hence the partition function will only contain contributions from graphs with closed loops. Let us consider the honeycomb lattice for simplicity, then there may be 0,1,2 or 3 powers of a given $\mathbf{s}(r_i)$. The trace over spins of $\mathbf{s}(r_i)^0$ will contribute a factor of 1, whereas, from $\mathrm{Tr}\,s_a(r)s_b(r)=\delta_{ab}$, instances of $\mathbf{s}(r_i)^2$ lead to a factor of n for each closed loop in the graph. The partition function is therefore equivalent to

$$Z_{O(n)}=\sum_G x^l n^v\,, \tag{2}$$

where l is the total length of all the loops in a given graph, v is the number of closed loops and the trace is now over all graphs of closed, non-intersecting loops. This form of the partition function may be used for general values of n, whereas equation (1) is applicable only to positive, integer n. There exists a critical point in the theory, at $x=x_c$, where the mean loop length diverges and the model is supposed to become conformally invariant. In order to formulate a field theory, the non-local factors of n must be made local. This can be done by assigning orientations to the loops (either clockwise or anticlockwise) and inserting local factors of $e^{i\chi}$ at each vertex where the curve turns to the right and $e^{-i\chi}$ where the curve turns to the left. Summing over the two possible orientations for each loop leads to a contribution of $e^{6i\chi}+e^{-6i\chi}$ from each closed loop on the honeycomb lattice, where closed loops have a difference in the number of left and right turns of six with the sign depending on the orientation of the loop. In order to obtain the desired factor of n for each closed loop, χ must therefore be chosen such that

$$e^{6i\chi}+e^{-6i\chi}=2\cos 6\chi=n\,. \tag{3}$$

This may now be transformed into a height model. The height variables are taken to be integer multiples of π and are assigned to sites on the dual lattice, such that the loops are contours of the landscape. Crossing a loop running from left to right leads to a decrease in the height of π, whilst crossing a loop running from right to left leads to an increase in the height of π. A given configuration of the heights corresponds to a unique graph of orientated loops. The assumption of the Coulomb gas is that this height model flows under the renormalisation group (RG) into a free field theory with action

$$S[\,h(\mathbf{r})]=\frac{g}{4\pi}\int(\partial h(\mathbf{r}))^2d^2\mathbf{r}\,,$$

for some $g(n)$. There is an additional subtlety regarding topology, which may be seen by considering the model defined on the cylinder. On a cylinder of circumference L, the loops wrapping around the circumference are counted incorrectly

(for $n \neq 2$). Loops wrapping around have the same number of left turns as right turns. These loops are therefore not counted correctly. The situation is remedied by placing a vertex operator (also known as an electric charge) $e^{i6\chi h/\pi}$ at one end of the cylinder and an operator $e^{-i6\chi h/\pi}$ at the other end. Let these points be $\pm w/2$. If there is a single loop wrapping around the cylinder, then there will be a height difference of π between the ends, the sign of which depends on the orientation of the loop. Summing over the two possibilities for the orientation then leads to the required contribution of $e^{i6\chi} + e^{-i6\chi}$. These charges at the ends of the cylinder lead to a modification of the partition function to

$$\begin{aligned} Z_{\text{Coulomb Gas}} &= Z_{\text{ffc}} \langle e^{-i\frac{6\chi}{\pi}h(-w/2)} e^{i\frac{6\chi}{\pi}h(w/2)} \rangle_{\text{ffc}} \\ &= Z_{\text{ffc}} \Big| \frac{L^2}{2\pi^2}(\cosh(\frac{2\pi w}{L}) - 1) \Big|^{(6\chi)^2/2\pi^2 g} \\ &= Z_{\text{ffc}} \Big| \frac{L^2}{4\pi^2} e^{2\pi w/L} \Big|^{(6\chi)^2/2\pi^2 g} , \end{aligned}$$

where the abbreviation ffc signifies that the above expectation values and partition functions are for the free field theory on the cylinder and where we have used $w \gg L$. The free energy per unit length on the cylinder for this Coulomb gas (CG) partition function may be calculated as

$$\frac{F_{CG}}{w} = -\frac{\ln(Z_{CG})}{w} = -\frac{\pi c}{6L} .$$

This is the form of the free energy on the cylinder obtained via the Schwartzian derivative from a theory on the plane with central charge c [5] given by

$$c = 1 - \frac{6}{g}\Big(\frac{6\chi}{\pi}\Big)^2 , \tag{4}$$

where the first term comes from the known behaviour of Z_{ffc} [5]. All that remains is to fix the value of g, which may be obtained from the following argument: adding a term $-\lambda \int \cos(2h) d^2 r$ to the action should not affect the critical behaviour, since the height variables were defined to be integer multiples of π and $\cos(2m\pi) = 1$ for all integer m. Hence, this term should be marginal under renormalisation group flow and therefore must have scaling dimension 2. This determines [8]

$$g = 1 - 6\chi/\pi . \tag{5}$$

3 Twist operators

In the loop (Coulomb) gas picture of the $O(n)$ model, loops wrapping around the cylinder may be counted with a weight n' different from n by placing additional charges $e^{\pm 6ih(\chi' - \chi)/\pi}$ at the ends of the cylinder, with χ' chosen according to equation (3). The scaling dimension of these operators may be obtained from their two point correlation function in the loop gas ensemble. Placing the charges at $\pm s/2$,

$$\lim_{w \gg L} \frac{\langle e^{-i\frac{6\chi}{\pi}h(-w/2)} e^{-i\frac{6(\chi'-\chi)}{\pi}\frac{h}{\pi}(-s/2)} e^{i\frac{6(\chi'-\chi)}{\pi}h(s/2)} e^{i\frac{6\chi}{\pi}h(w/2)} \rangle_{\text{free-field, cyl.}}}{\langle e^{-i\frac{6\chi}{\pi}h(-w/2)} e^{i\frac{6\chi}{\pi}h(w/2)} \rangle_{\text{free-field, cyl.}}} .$$

These free field expectation values may be calculated explicitly since they are Gaussian moments. Then, in the limit $s \gg L$, the two point function is seen to be

$$\left|\frac{L^2}{4\pi^2} e^{2\pi w/L}\right|^{-6^2(\chi'^2-\chi^2)/2\pi^2 g} .$$

This expression is of the general form of a two point function of operators on the cylinder with scaling dimension [9]

$$x(n, n') = 6^2(\chi'^2 - \chi^2)/2\pi^2 g \,. \tag{6}$$

A particular choice of interest is $n' = -n$, for which the scaling dimension is $x(n, -n) = 3/2g - 1$; this is the scaling dimension of $\phi_{1,2}$ operators with level two null states in the Kac classification:

$$x_{r,s} = \frac{(rg - s)^2 - (g - 1)^2}{2g} .$$

These correspond, in string theory language, to operators which insert orbifold points corresponding to the global symmetry $\mathbf{s} \to -\mathbf{s}$ of the hamiltonian and are known as twist operators. They will be the focus of this paper. Their correlation functions may be considered in geometries other than the cylinder. For example, on the Riemann sphere, the two point function of such operators is fixed by scale invariance to be

$$\langle \phi(z_1, \overline{z_1})\phi(z_2, \overline{z_2})\rangle_{\mathrm{CFT}} = \left|\frac{z_1 - z_2}{a}\right|^{2-3/g(n)} , \tag{7}$$

where we have inserted explicit factors of the lattice spacing, a, so as to make ϕ dimensionless. The two point correlation function in the loop gas may also be calculated on the sphere using the ensemble of equation (2). Each loop separating the points $(z_1, \overline{z_1})$ and $(z_2, \overline{z_2})$ will be counted with weight $-n$ rather than n, hence graphs in G will be weighted with an additional factor of (-1) to the power of the number of loops separating the two points. Graphs in G with an odd number of loops separating point $(z_1, \overline{z_1})$ from point $(z_2, \overline{z_2})$ will be weighted by an additional factor of $(-1)^{\mathrm{odd}} = -1$, whilst those with an even number of loops will be unaffected, since $(-1)^{\mathrm{even}} = 1$. These two different types of graphs can be separated in the sum over graphs as follows

$$\langle \phi(z_1, \overline{z_1})\phi(z_2, \overline{z_2})\rangle_{\text{loop gas}} = \Big[\sum_{G_{\mathrm{odd}}(z_1,z_2)} (-1) x_c^{\mathrm{l}} n^{\mathrm{v}} + \sum_{G_{\mathrm{even}}(z_1,z_2)} x_c^{\mathrm{l}} n^{\mathrm{v}}\Big] \Big/ Z_{\text{loop gas}} , \tag{8}$$

where $G_{\mathrm{odd}}(z_1, z_2)$ is the set of graphs of closed, non-intersecting loops with an odd number of loops separating point $(z_1, \overline{z_1})$ and $(z_2, \overline{z_2})$ and the $G_{\mathrm{even}}(z_1, z_2)$ is the set with an even number of loops separating the two points. This two point correlation function has a natural interpretation in terms of a defect line joining points $(z_1, \overline{z_1})$ and $(z_2, \overline{z_2})$. If N_{12} is the number of times loops cross the defect line in a given graph, then equation (8) may be rewritten as

$$\begin{aligned}\langle \phi(z_1, \overline{z_1})\phi(z_2, \overline{z_2})\rangle_{\text{loop gas}} &= \sum_G (-1)^{N_{12}} x_c^{\mathrm{l}} n^{\mathrm{m}} \Big/ Z_{\text{loop gas}} \\ &= \langle (-1)^{N_{12}}\rangle_{\text{loop gas}} .\end{aligned} \tag{9}$$

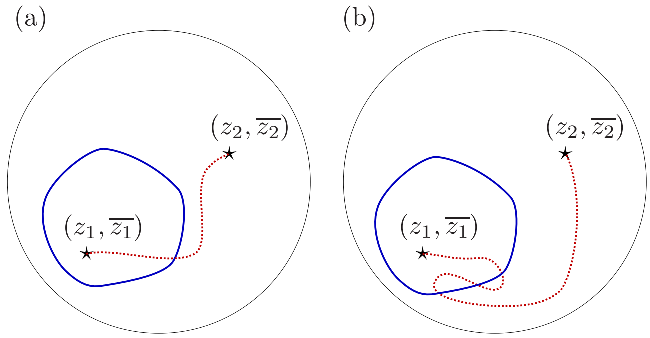

Figure 3: A graph on the Riemann sphere belonging to $\{G_{\text{odd}}\}$ with one loop (solid line) and two choices of path for the defect line (dashed line). In (a) $N_{12} = 1$ so that $(-1)^{N_{12}} = -1$. In (b) $N_{12} = 3$ so that $(-1)^{N_{12}} = -1$ also. Any choice of path for the defect line will lead to $(-1)^{N_{12}} = -1$ since N_{12} is always odd.

The defect line can take any path between the two points because, for a given loop configuration, $(-1)^{N_{12}}$ will be the same for all paths. This can be best understood with reference to figure 3. Combining equations (7) and (9), we find

$$\langle(-1)^{N_{12}}\rangle_{\text{loop gas}} = \left|\frac{z_1 - z_2}{a}\right|^{2-3/g(n)} . \tag{10}$$

3.1 The small n limit

Self-avoiding loops correspond to $g = 3/2$, which is the dilute phase of the $n \to 0$ limit of the $O(n)$ model. The results of section 3 are summarized by equation (10), which for $n \ll 1$ may be expanded in powers of n. The order n^1 term in the loop gas expansion (the left hand side of the equation) comes from graphs with a single loop. By equating this with the order n^1 term of the right hand side, we shall demonstrate a property of the sum over graphs with a single loop.

The left hand side of equation (10) for $n \ll 1$ becomes

$$\Big(\sum_{G_0}(-1)^{N_{12}}x_c^l n^0 + \sum_{G_1}(-1)^{N_{12}}x_c^l n^1\Big)\Big(\sum_{G_0}x_c^l n^0 + \sum_{G_1}x_c^l n^1\Big)^{-1} + O(n^2)\,, \tag{11}$$

where G_0 is the set of all graphs with no loops and G_1 is the set of all graphs with one loop, hence their respective powers of n^0 and n^1. G_0 contains a single graph with no edges, hence the first term in both brackets is equal to 1. The set G_1 contains graphs with only a single loop. This set is composed of two subsets with no overlap, the set $G_{1,z_1|z_2}$ with a single loop separating the point $(z_1, \overline{z_1})$ from $(z_2, \overline{z_2})$ and the set $G_{1,z_1 z_2|}$ which contains no loops separating the points. It should be noted that a single loop surrounding the point $(z_1, \overline{z_1})$ is classified in the same group as a loop around $(z_2, \overline{z_2})$ since on the Riemann sphere one can be continuously deformed into the other. Both have an odd number of intersections with a defect line between the two points, hence $(-1)^{N_{12}} = -1$.

For all graphs belonging to $G_{1,z_1z_2|}$, it may be seen that $(-1)^{N_{12}}=1$. This is shown in figure 4.

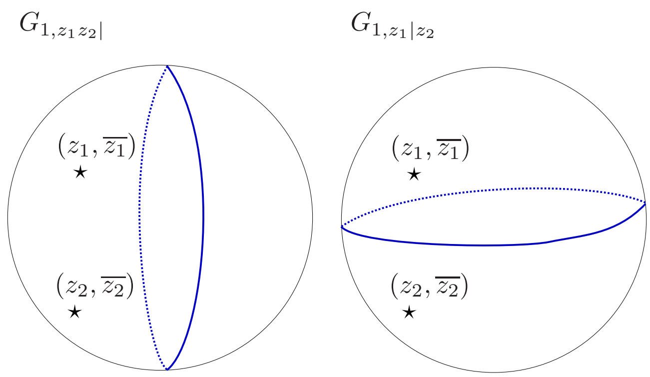

Figure 4: An example of a graph belonging to $G_{1,z_1z_2|}$ and one belonging to $G_{1,z_1|z_2}$ for a choice of $(z_1,\overline{z_1})$ and $(z_2,\overline{z_2})$ on the Riemann sphere. Notice that a loop in $G_{1,z_1|z_2}$ can be thought of as being either around $(z_1,\overline{z_1})$ or around $(z_2,\overline{z_2})$. Similarly, a loop in $G_{1,z_1z_2|}$ can be thought of as being around both points or neither.

Equation (11) can therefore be rewritten as

$$\Big(1+\sum_{G_{1,z_1|z_2}}(-1)x_c^l n^1+\sum_{G_{1,z_1z_2|}}x_c^l n^1\Big)\Big(1+\sum_{G_{1,z_1|z_2}}x_c^l n^1+\sum_{G_{1,z_1z_2|}}x_c^l n^1\Big)^{-1}+O(n^2)$$
$$=1-2n\sum_{G_{1,z_1|z_2}}x_c^l+O(n^2)\,. \tag{12}$$

Equations (6), (3) and (5) may be combined and then expanded in powers of n to find the following form for the scaling dimension of the twist operators in the small n limit:

$$x(n)=-1+3/2g(n)=n/3\pi+O(n^2)\,.$$

Therefore, the expansion of the right hand side of equation (10) is

$$\left|\frac{z_1-z_2}{a}\right|^{-2n/3\pi}+O(n^2)=1-\frac{2n}{3\pi}\ln\left|\frac{z_1-z_2}{a}\right|+O(n^2) \tag{13}$$

Hence, from comparing the coefficients of n^1 in equations (12) and (13), we may conclude that

$$\sum_{G_{1,z_1|z_2}}x_c^l=\frac{1}{3\pi}\ln\left|\frac{z_1-z_2}{a}\right|\,. \tag{14}$$

In words, for a given pair of points $(z_1,\overline{z_1})$ and $(z_2,\overline{z_2})$, the number of weighted loops which separate the two points, weighted by x_c^l, diverges as $(1/3\pi)\ln|1/a|$ in the limit of the lattice spacing tending to zero. The cause of this is the diverging contribution from vanishingly small loops in the continuum limit. The factor x_c^l may be thought of as the measure, hence this weighted number of loops

is equivalent to the expected number of loops around one point. It may be noted that the coefficient $1/3\pi$ differs from that seen on the annulus with shrinking internal radius. In the case of the annulus with vanishingly small modulus [10] a coefficient of $1/6\pi$ is seen, because there are only diverging contributions from small loops around one point, as opposed to the plane where small loops around both points contribute.

4 The four point correlation function of twist operators from conformal field theory

Twist operators were introduced in section 3 as the operators responsible for counting loops with weight $-n$ rather than n in the loop gas picture of the $O(n)$ model. Their scaling dimension was calculated and it was seen that it corresponds to the scaling dimension of operators with null states at level two. Conformal field theory may be used to derive a set of partial differential equations satisfied by the correlation functions of such null state operators. In the case of the four point function, these differential equations may be solved analytically, as will be seen in this section. To make contact with the language of Schramm (stochastic)-Loewner Evolution (SLE), g is dropped in favour of the parameter κ used in SLE [11]. The two are related by the formula $\kappa = 4/g$.

It was shown in section 3 that the twist operators have a null state at level two [5]. This null state is itself a highest weight state; it is annihilated by all raising operators, L_n with $n > 0$. This is not the same as saying that the null state decouples from all other states in the theory, as it does in unitary theories. That said, it can be shown by a modular invariance argument that physical results are seen only if the null state decouples on the torus. This is because, barring unforeseen cancellations, the nondecoupling of the null states would lead to a density of states at high energies corresponding to $c = 1$ rather than $c < 1$ [12]. We shall therefore set the null state to zero in this theory and use the resulting partial differential equations for the correlation functions of the twist operators. In the complex plane, the four point function satisfies the following set of partial differential equations (for $j = 1, 2, 3, 4$) [6]:

$$\Big[\partial_{z_j}^2 - \frac{\kappa}{4}\sum_{i\neq j}\frac{h_2}{(z_i-z_j)^2} - \frac{\partial_{z_i}}{z_i-z_j}\Big]\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle = 0 \tag{15}$$

$$\Big[\partial_{\overline{z_j}}^2 - \frac{\kappa}{4}\sum_{i\neq j}\frac{\overline{h_2}}{(\overline{z_i}-\overline{z_j})^2} - \frac{\partial_{\overline{z_i}}}{\overline{z_i}-\overline{z_j}}\Big]\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle = 0. \tag{16}$$

In terms of $z_{ij} \equiv z_i - z_j$, the cross ratios are defined as

$$\eta \equiv \frac{z_{12}z_{34}}{z_{13}z_{24}} \qquad\qquad \overline{\eta} \equiv \frac{\overline{z_{12}}\,\overline{z_{34}}}{\overline{z_{13}}\,\overline{z_{24}}}\,.$$

These cross ratios are invariant under global conformal transformations. The

partial differential equations above have the solutions

$$\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle_{\rm CFT} = \Big|\frac{z_{13}z_{24}a^2}{z_{12}z_{34}z_{23}z_{14}}\Big|^{4h_2} A(\kappa)\xi(\eta,\overline{\eta},\kappa)\,, \tag{17}$$

where $G(\eta,\overline{\eta},\kappa)$ is

$$\begin{aligned}\xi(\eta,\overline{\eta},\kappa) &= |{}_2F_1(1-\frac{\kappa}{4},2-\frac{3\kappa}{4};2-\frac{\kappa}{2};\eta)|^2\\ &\quad +B(\kappa)|\eta(1-\eta)|^{2h_3}|{}_2F_1(\frac{\kappa}{4},\frac{3\kappa}{4}-1;\frac{\kappa}{2};\eta)|^2\,,\end{aligned}$$

$h_2 = 3\kappa/16 - 1/2$ and $h_3 = \kappa/2 - 1$ is the value of the scaling dimension of an operator with a null state at level three. The form of $B(\kappa)$ can be derived by requiring consistency under the permutation of the labels $(z_i,\overline{z_i})$ and is found to be [13, 14]

$$B(\kappa) = \frac{[\Gamma(1-\frac{\kappa}{4})^2\Gamma(\frac{\kappa}{4})^2 - \Gamma(2-\frac{\kappa}{2})^2\Gamma(\frac{\kappa}{2}-1)^2]\Gamma(\frac{3\kappa}{4}-1)^2}{\Gamma(\frac{\kappa}{2})^2\Gamma(\frac{\kappa}{2}-1)^2\Gamma(1-\frac{\kappa}{4})^2}\,.$$

The form of $A(\kappa)$ is dependent on the normalisation of the two point function since, as $\eta\to 0$, the four point function becomes

$$\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle_{\rm CFT} \to A(\kappa)\Big|\frac{z_{12}z_{34}}{a^2}\Big|^{-4h_2}\,.$$

4.1 Interpretation of the four point correlation function

Recall from section 3 that the correlation function of two ϕ operators has an interpretation in terms of a defect line in the loop gas ensemble. The operators are the source and sink of the defect line, which can take any path between the two points. Each graph in the sum is weighted by an additional factor $(-1)^{N_{12}}$ where N_{12} is the number of intersections of loops in that graph with the defect line. There is a similar interpretation of the four point function in the loop gas ensemble, but now there are two defect lines. The four points are split into two pairs and defect lines run between the points in each pair. For the choice of pairing $(z_1,\overline{z_1})$ with $(z_2,\overline{z_2})$ and $(z_3,\overline{z_3})$ with $(z_4,\overline{z_4})$, the four point function can be seen to correspond to the following expectation value in the loop gas picture

$$\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle_{\text{loop gas}} = \langle(-1)^{N_{12}}(-1)^{N_{34}}\rangle_{\text{loop gas}}\,, \tag{18}$$

where N_{12} is the number of crossings of loops across a defect line from $(z_1,\overline{z_1})$ to $(z_2,\overline{z_2})$ and N_{34} is similarly defined. There are three ways in which the pairs may be chosen, but each choice leads to the same set of weights for the graphs in the partition function. The reason for this may be seen in figure 5. Thus, the four point function may also be considered to be the expectation value in the loop gas ensemble of $(-1)^{N_{13}}(-1)^{N_{24}}$ or $(-1)^{N_{14}}(-1)^{N_{23}}$. Comparing equations (17) and (18) leads to the main result of this section

$$\langle(-1)^{N_{12}}(-1)^{N_{34}}\rangle_{\text{loop gas}} = \Big|\frac{z_{13}z_{24}a^2}{z_{12}z_{34}z_{23}z_{14}}\Big|^{4h_2} A(\kappa)\xi(\eta,\overline{\eta},\kappa)\,, \tag{19}$$

where $\xi(\eta,\overline{\eta},\kappa)$ is defined below equation (17) in section 4.

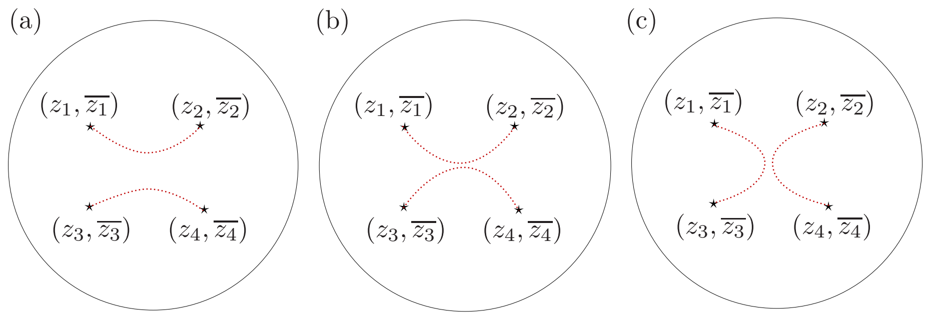

Figure 5: (a) shows the choice of defect lines described in the text. As the defect lines may take any path between their endpoints, they may be re-routed to pass infinitesimally close to each other, as in (b). Since a loop crossing a defect line leads to an additional weight (-1) in the loop gas partition function, a pair of defect lines leads to a factor $(-1)^2 = 1$ and hence makes no contribution to the weight of any graph. Hence, the defect lines shown in (b) are equivalent to the choice in (c). A similar construction may be used to show the equivalence to a pair of defect lines drawn from $(z_1, \overline{z_1})$ to $(z_2, \overline{z_2})$ and from $(z_3, \overline{z_3})$ to $(z_4, \overline{z_4})$. Note that this feature is a result of the choice $n' = -n$ in section 3.

4.2 Example: the Ising model

Boundaries between Ising spin clusters correspond to the loops of the $O(n)$ model at $n = 1$, or $\kappa = 3$. The twist operators in this case are equivalent to magnetisation operators because the parity of the number of loops separating two spins depends on whether they are parallel or anti-parallel. That is to say that

$$\begin{aligned}\langle(-1)^{N_{12}}\rangle_{\text{loop gas}} &= \left|\frac{a}{z_1 - z_2}\right|^{4h_2} \\ &= \langle s(z_1, \overline{z_1}) s(z_2, \overline{z_2})\rangle_{\text{Ising}}\,.\end{aligned}$$

Similarly, the four point correlation function of the twist operators is the four point function of the magnetisation operators. The following function is the result of substituting $\kappa = 3$ into the results of section 4

$$\langle\phi(z_1, \overline{z_1})\phi(z_2, \overline{z_2})\phi(z_3, \overline{z_3})\phi(z_4, \overline{z_4})\rangle = \frac{A(3)}{2}\left|\frac{z_{13}z_{24}a^2}{z_{12}z_{34}z_{23}z_{14}}\right|^{1/4}\left(\,|1+\sqrt{\eta}\,| + |1-\sqrt{\eta}\,|\,\right). \tag{20}$$

Equation (20) is the well known four point correlation of spin operators in the Ising model at criticality [6].

5 The small n limit of the four point function

In section 3, we examined the small n limit of the two point function of twist operators, equation (10). We found an exact expression for the μ-mass of loops surrounding one of the two points. This result is summarised by equation (14).

A similar expansion to order n^1 of the four point correlation function (equation 19) yields information about the configurations of a single self-avoiding loop around four points.

From the equations in section 4, the following small n expansions may be deduced

$$
\begin{aligned}
\kappa &= \frac{8}{3} + \frac{8}{9\pi} n + O(n^2) \\
h_2 &= \frac{n}{6\pi} + O(n^2) \\
h_3 &= \frac{1}{3} + \frac{4}{9\pi} n + O(n^2) \\
B &= \frac{8(2)^{1/3}\pi}{3\sqrt{3}\Gamma(\frac{1}{6})^2\Gamma(\frac{4}{3})^2} n + O(n^2)\,.
\end{aligned}
$$

The expansion of $A(n)$ around $n=0$ is $A(n) = 1 + \varrho n + O(n^2)$ where ϱ is a constant independent of n, since the correlation function should tend to 1 as $n \to 0$. The Taylor series expansion of the right hand side of equation (19) is then

$$
\begin{aligned}
&\langle \phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle_{\mathrm{CFT}} = \\
&\quad 1 + n\Big[\varrho - \frac{2}{3\pi}\ln|\eta\, z_{23} z_{14} a^{-2}| - \frac{1}{3\pi}\Big(\eta\, {}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta) + \overline{\eta}\, {}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\overline{\eta})\Big) \\
&\qquad + \frac{8(2)^{1/3}\pi}{3\sqrt{3}\Gamma(\frac{1}{6})^2\Gamma(\frac{4}{3})^2}|\eta(1-\eta)|^{2/3}|\,{}_2F_1(\frac{2}{3},1;\frac{4}{3};\eta)|^2\Big] + O(n^2)\,. \qquad (21)
\end{aligned}
$$

The left hand side of equation (19) may also be expanded in powers of n. The graphs up to order n^1 in G may be split into the graph with no loops (n^0) and eight distinct sets of graphs with a single loop, as will be shown in the next section.

5.1 The configurations of a single loop around four points

In this section, we expand the left hand side of equation (19) for $n \ll 1$ to order n^1. The coefficient of n^1 may then be equated with the result for expansion of the right hand side, described in the previous section and leading to equation (21). Recall that the partition function for the loop gas expansion is equation (2). The sum may be decomposed into a term of order n^0 coming from the graph with no loops (hence the zeroth power of n) and graphs with a single loop weighted by n^1. Other graphs will contribute terms of order n^2 or smaller. The graphs with a single loop on the Riemann sphere may be further split up into eight distinct subsets, which are shown in figure 6. Each configuration has a unique set of $(-1)^{N_{ij}}$ where $i \neq j$ and $i,j \in \{1,2,3,4\}$.

Let L be defined as

$$
L \equiv \langle (-1)^{N_{12}}(-1)^{N_{34}}\rangle_{\text{loop gas}} = \frac{\sum_G (-1)^{N_{12}}(-1)^{N_{34}} n^{\#\text{loops}} x_c^{\text{total length of loops}}}{\sum_G n^{\#\text{loops}} x_c^{\text{total length of loops}}}\,.
$$

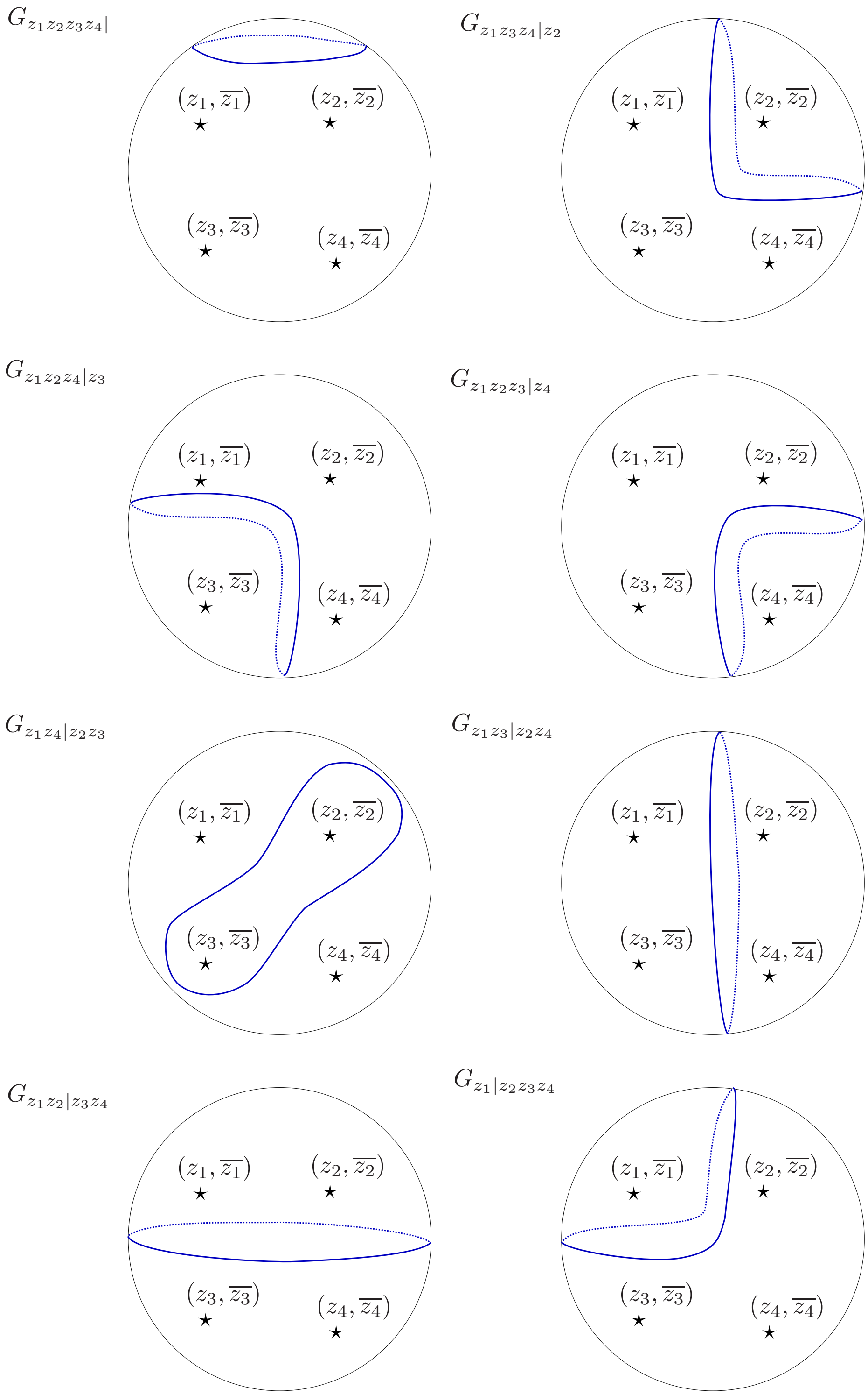

Figure 6: The eight distinct configurations of a loop around a given set of four points on the Riemann sphere.

The numerator of L can be calculated as

$$
\begin{aligned}
L_{\text{numerator}} = 1 + n\Big[&\sum_{G_{z_1 z_2 z_3 z_4|}} (1)(1)x_c^l + \sum_{G_{z_1 z_2 z_3|z_4}} (1)(-1)x_c^l + \sum_{G_{z_1 z_2 z_4|z_3}} (1)(-1)x_c^l \\
&+ \sum_{G_{z_1 z_3 z_4|z_2}} (-1)(1)x_c^l + \sum_{G_{z_1 z_2|z_3 z_4}} (1)(1)x_c^l + \sum_{G_{z_1 z_3|z_2 z_4}} (-1)(-1)x_c^l \\
&+ \sum_{G_{z_1 z_4|z_2 z_3}} (-1)(-1)x_c^l + \sum_{G_{z_1|z_2 z_3 z_4}} (-1)(1)x_c^l \Big] + O(n^2)\,.
\end{aligned}
$$

The denominator takes the same form as the expression above, but without the factors of (1) and (−1). For ease of notation, define

$$
W_{z_1 z_2 z_3 z_4|} = \sum_{G_{z_1 z_2 z_3 z_4|}} x_c^l\,,
$$

and similar expressions for the other graphs in figure 6. The elements of $\{W_i\}$ are sums over all loops of a single configuration type i, weighted by x_c to the power of their length. They therefore represent the μ-masses of loops of configuration i. It may then be seen that

$$
L = 1 - 2n\Big[W_{z_1 z_2 z_3|z_4} + W_{z_1 z_2 z_4|z_3} + W_{z_1 z_3 z_4|z_2} + W_{z_1|z_2 z_3 z_4}\Big] + O(n^2) \tag{22}
$$

The two point functions $C_{\{ij\}} \equiv \langle \phi(z_i, \overline{z_i})\phi(z_j, \overline{z_j})\rangle = \langle (-1)^{N_{ij}}\rangle$ and products of pairs of two point functions can be expanded in terms of the $\{W_i\}$ also:

$$
\begin{aligned}
C_{12}C_{34} = 1 - 2n\big(&W_{z_1 z_3 z_4|z_2} + W_{z_1 z_2 z_4|z_3} + W_{z_1 z_2 z_3|z_4} \\
&+ W_{z_1|z_2 z_3 z_4} + 2W_{z_1 z_4|z_2 z_3} + 2W_{z_1 z_3|z_2 z_4}\big) + O(n^2) && (23)\\
C_{13}C_{24} = 1 - 2n\big(&W_{z_1 z_3 z_4|z_2} + W_{z_1 z_2 z_4|z_3} + W_{z_1 z_2 z_3|z_4} \\
&+ W_{z_1|z_2 z_3 z_4} + 2W_{z_1 z_4|z_2 z_3} + 2W_{z_1 z_2|z_3 z_4}\big) + O(n^2) && (24)\\
C_{14}C_{23} = 1 - 2n\big(&W_{z_1 z_3 z_4|z_2} + W_{z_1 z_2 z_4|z_3} + W_{z_1 z_2 z_3|z_4} \\
&+ W_{z_1|z_2 z_3 z_4} + 2W_{z_1 z_2|z_3 z_4} + 2W_{z_1 z_3|z_2 z_4}\big) + O(n^2)\,. && (25)
\end{aligned}
$$

Thus, it can be seen from equations (22),(23),(24) and (25) that a subset $\{W_c\}$ of the $\{W_i\}$ can be expressed in terms of partly-connected four point functions:

$$
\begin{aligned}
W_{z_1 z_4|z_2 z_3} &= \frac{1}{8n}(L - C_{12}C_{34} - C_{13}C_{24} + C_{14}C_{23}) + O(n) && (26)\\
W_{z_1 z_3|z_2 z_4} &= \frac{1}{8n}(L - C_{12}C_{34} + C_{13}C_{24} - C_{14}C_{23}) + O(n) && (27)\\
W_{z_1 z_2|z_3 z_4} &= \frac{1}{8n}(L + C_{12}C_{34} - C_{13}C_{24} - C_{14}C_{23}) + O(n)\,. && (28)
\end{aligned}
$$

These $\{W_c\}$ correspond to graphs with a loop winding around two of the four points and hence do not include contributions from vanishingly small loops in the continuum limit. By comparison with the small n expansions of L (section 5) and $C_{\{ij\}}$ (section 3.1 with the same choice for the normalisation of the twist operator as in the four point function) from conformal field theory, these weights

are the following functions of $\eta, \overline{\eta}$

$$W_{z_1 z_4|z_2 z_3} = \frac{-1}{6\pi} \ln|1-\eta| + q(\eta,\overline{\eta}) + O(n) \tag{29}$$

$$W_{z_1 z_3|z_2 z_4} = q(\eta,\overline{\eta}) + O(n) \tag{30}$$

$$W_{z_1 z_2|z_3 z_4} = \frac{-1}{6\pi} \ln|\eta| + q(\eta,\overline{\eta}) + O(n)\,, \tag{31}$$

where

$$q(\eta,\overline{\eta}) = \frac{-1}{24\pi}\Big(\eta\, {}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta) + \overline{\eta}\, {}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\overline{\eta})\Big) + \frac{2^{1/3}\pi}{3\sqrt{3}\Gamma(\frac{1}{6})^2\Gamma(\frac{4}{3})^2}|\eta(1-\eta)|^{2/3}|\,{}_2F_1(\frac{2}{3},1,\frac{4}{3},\eta)|^2\,.$$

Note that the W_c above are finite in the continuum limit of vanishing lattice spacing. In fact, the expressions in equations (29),(30) and (31) are independent of a. They are finite and non-zero in the limit $n \to 0$ and are invariant under conformal transformations, being functions only of the cross ratios. These expressions for the μ-masses $\{W_c\}$ constitute one of the main results of this paper.

5.2 The central charge

It is a standard result of conformal field theory that, given the explicit form of any four-point function, the central charge c may be determined. This is because the operator product expansion (OPE) of any scalar primary operator ϕ with itself takes the form

$$\phi(z_1,\bar{z}_1)\phi(z_2,\bar{z}_2) = |z_{12}|^{-4h}\left(1 + \cdots + \frac{2h}{c}T(z_1) + \cdots\right) \tag{32}$$

where T is the holomorphic component of the stress tensor. The coefficient follows from consideration of the limit $z_{12} \to 0$ in the three point function $\langle T(z)\phi(z_1)\phi(z_2)\rangle \propto 2h$ and the two-point function $\langle T(z)T(z_1)\rangle = (c/2)(z-z_1)^{-4}$.

If this result is applied to the explicit expression (17), we find that the coefficient of η^2 is

$$h_2(2h_2+1) + \frac{2h_2(1-\frac{\kappa}{4})(2-\frac{3\kappa}{4})}{2-\frac{\kappa}{2}} + \frac{(1-\frac{\kappa}{4})(2-\frac{\kappa}{4})(2-\frac{3\kappa}{4})(3-\frac{3\kappa}{4})}{2(2-\frac{\kappa}{2})(2-\frac{\kappa}{2})}\,,$$

and so obtain the known expression for the central charge in terms of κ

$$c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}\,. \tag{33}$$

This result is of interest in the limit $n \to 0$ for the light it sheds on the interpretation of the stress tensor T in terms of an observable of a random curve given by Doyon, Riva and Cardy [7]. In that paper it was shown that, for any measure on simple random curves which satisfies conformal restriction, one may

identify $T(z)$ as being proportional to the spin-2 angular Fourier component of the probability that the curve intersects a small line segment of length ϵ centred at z. While in that paper the focus was on curves which connect two points on the boundary of a simply connected domain, described in the case when conformal restriction holds by $\mathrm{SLE}_{8/3}$, the theorem should equally well apply to Werner's measure on self-avoiding loops, if suitably re-interpreted in terms of μ-masses rather than probabilities. However our results in the present paper do not directly apply to the intersection with a line segment, but rather to the event that the loop passes (or does not pass) between pairs of points (which, however, may be taken to mark the ends of the line segment). Nevertheless, one might expect these to differ only by a constant of proportionality, and indeed our results in this paper support this, and suggest that the two-point function of the object T introduced in Ref. [7] is indeed given by the central charge for small n as expected.

Indeed, let us consider the quantity $W_{z_1 z_4|z_2 z_3}$ as given in equation 29. This is the μ-mass of loops which separate (z_1, z_4) from (z_2, z_3). In the limit $z_{12} \to 0$, writing $z_{12} = \epsilon e^{i\theta_{12}}$, we can define

$$V(z_1; z_3, z_4) = 5 \lim_{\epsilon \to 0} \epsilon^{-2} \int d\theta_{12} e^{-2i\theta_{12}} W_{z_1 z_4|z_2 z_3} \,, \tag{34}$$

where the numerical prefactor is $\lim_{n\to 0}(c/2h_2)$. Our result equation 17 then implies that V has the same dependence on its arguments as does the $\mathrm{O}(n)$ term in the CFT correlation function

$$\langle T(z_1) \phi_{1,2}(z_3, \bar{z}_3) \phi_{1,2}(z_4, \bar{z}_4) \rangle \,. \tag{35}$$

This result generalises to other correlation functions, and implies that we may interpret the spin-2 component of the μ-mass of loops which pass between two nearby points as being, in some sense, the derivative $\widetilde{T}$ of the CFT stress tensor with respect to n at $n = 0$. With this definition we then find the result

$$\langle \widetilde{T}(z_1) \widetilde{T}(z_3) \rangle = \frac{c'(0)/2}{(z_1 - z_3)^4} \,, \tag{36}$$

as expected, where the left hand side is

$$5^2 \lim_{\epsilon \to 0} \epsilon^{-4} \int d\theta_{12} e^{-2i\theta_{12}} \int d\theta_{34} e^{-2i\theta_{34}} W_{z_1 z_4|z_2 z_3} \tag{37}$$

It would, of course, be important to establish this interpretation directly from the restriction property.

6 Twist operators in a simply connected domain

The model may also be considered in a simply connected domain. All such domains may be conformally transformed to the upper half plane, with the real axis being the boundary. We may therefore derive the results for the upper half plane. The two point function of twist operators in the presence of this boundary satisfies the same set of partial differential equations as the four point

function in the bulk, with $(z_3,\overline{z_3})$,$(z_4,\overline{z_4})$ assigned as the complex conjugates of the positions of the operators. This is a well known result in the theory of boundary conformal field theory [5] (BCFT) and follows from the condition of $T=\overline{T}$ on the boundary. The solution for the two point function is

$$\langle\phi_2(z_1)\phi_2(z_2)\rangle_{\rm BCFT}=\Big(\frac{(z_1-z_1^*)(z_2-z_2^*)a^2}{(z_1-z_2)(z_1^*-z_2^*)(z_2-z_1^*)(z_1-z_2^*)}\Big)^{3\kappa/8-1}\times$$
$$A(\kappa)\Big[{}_2F_1(1-\frac{\kappa}{4},2-\frac{3\kappa}{4};2-\frac{\kappa}{2};\eta)+B(\kappa)(-\eta(1-\eta))^{\kappa/2-1}\,{}_2F_1(\frac{\kappa}{4},\frac{3\kappa}{4}-1;\frac{\kappa}{2};\eta)\Big]\,, \tag{38}$$

where now there is only one cross-ratio

$$\eta=\frac{(z_1-z_2)(z_1^*-z_2^*)}{(z_1-z_1^*)(z_2-z_2^*)}\,.$$

$B(\kappa)$ can be determined by looking at the boundary conditions as the two points approach the boundary, which is $\eta\to-\infty$. Equation (38) may be analytically continued to large η:

$$\Big(\frac{(z_2-z_1^*)(z_2^*-z_1)}{a^2}\Big)^{1-\frac{3\kappa}{8}}A(\kappa)\times \tag{39}$$

$$\Big[(\frac{-1}{\eta})^{\frac{\kappa}{8}}\,{}_2F_1\big(1-\frac{\kappa}{4},\frac{\kappa}{4};\frac{\kappa}{2};\frac{1}{\eta}\big)\Big(\frac{\Gamma(2-\frac{\kappa}{2})\Gamma(1-\frac{\kappa}{2})}{\Gamma(2-\frac{3\kappa}{4})\Gamma(1-\frac{\kappa}{4})}+B\frac{\Gamma(\frac{\kappa}{2})\Gamma(1-\frac{\kappa}{2})}{\Gamma(\frac{\kappa}{4})\Gamma(1-\frac{\kappa}{4})}\Big)+$$
$$(-\eta)^{3\kappa/8-1}\,{}_2F_1\big(2-\frac{3\kappa}{4},\frac{4-\kappa}{4};2-\frac{\kappa}{2};\frac{1}{\eta}\big)\Big(\frac{\Gamma(2-\frac{\kappa}{2})\Gamma(\frac{\kappa}{2}-1)}{\Gamma(1-\frac{\kappa}{4})\Gamma(\frac{\kappa}{4})}+B\frac{\Gamma(\frac{\kappa}{2})\Gamma(\frac{\kappa}{2}-1)}{\Gamma(\frac{3\kappa}{4}-1)\Gamma(\frac{\kappa}{4})}\Big)\Big]\,. \tag{40}$$

The two natural choices of $B(\kappa)$ are those which pick out one or the other conformal block as the operators approach the boundary. In the limit $n\to0$, there are no loops in the loop gas picture so the correlation function should tend to unity for all η. This is only possible if the first term in the square brackets of equation (40) is not present. A (non-unique) choice of $B(\kappa)$ which satisfies this requirement is that which picks out the second conformal block only:

$$B(\kappa)=-\frac{\Gamma(2-\frac{\kappa}{2})\Gamma(1-\frac{\kappa}{2})\Gamma(\frac{\kappa}{4})\Gamma(1-\frac{\kappa}{4})}{\Gamma(\frac{\kappa}{2})\Gamma(1-\frac{\kappa}{2})\Gamma(2-\frac{3\kappa}{4})\Gamma(1-\frac{\kappa}{4})}\,.$$

It is important to note also that this boundary condition is not compatible with the vanishing of the four point function in the limit $\eta\to-\infty$. The form of $A(\kappa)$ is dependent on the choice of normalisation of the twist operators. This can be seen from the $\eta\to0$ limit of equation (38):

$$\lim_{\eta\to0}\langle\phi_2(z_1)\phi_2(z_2)\rangle=A(\kappa)\Big(\frac{(z_1-z_2)(z_1^*-z_2^*)}{a^2}\Big)^{1-3\kappa/8}\,.$$

$A(\kappa)$ must be of the form $A=1+\sigma n+O(n^2)$ due to the constraint that the four point function tends to unity as $n\to0$. For small n, the correlation function may be expanded as

$$\langle\phi_2(z_1)\phi_2(z_2)\rangle_{\rm BCFT}=1+n\Big[\frac{1}{3\pi}\ln(\mu)-\frac{1}{3\pi}\eta\,{}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta)$$
$$+\frac{2\Gamma(\frac{2}{3})^2}{3\pi\Gamma(\frac{4}{3})}(-\eta(1-\eta))^{\frac{1}{3}}\,{}_2F_1(\frac{2}{3},1;\frac{4}{3};\eta)+\sigma\Big]+O(n^2)\,,$$

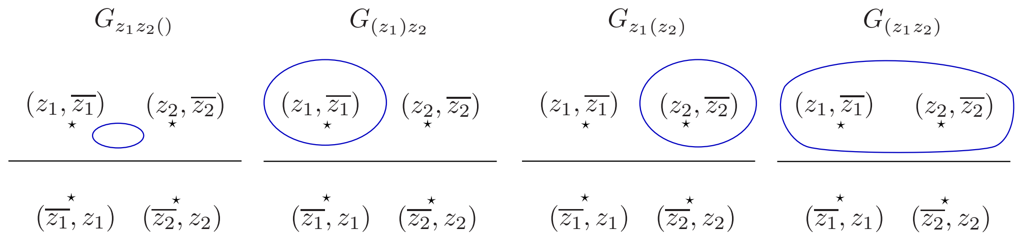

Figure 7: The four types of configuration of a single loop around two points in the presence of a boundary. The complex conjugate points are included only for ease of calculation of quantities such as $(-1)^{N_{1,1^*}}$ (see text).

where

$$\mu = \frac{(z_1 - z_1^*)(z_2 - z_2^*)a^2}{(z_1 - z_2)(z_1^* - z_2^*)(z_2 - z_1^*)(z_1 - z_2^*)}\,.$$

In analogy with the interpretation of the four point function in the bulk in the loop gas picture described in section 4.1, the following is the interpretation of the two point function in the presence of a boundary

$$\langle(-1)^{N_{1,1^*}}(-1)^{N_{2,2^*}}\rangle_{\text{loop gas}} = \langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\rangle_{\text{BCFT}}\,. \tag{41}$$

The left hand side is an expectation value in the ensemble of lattice loops, as before. $N_{1,1^*}$ is the number of times loops cross a defect line from z_1 and z_1^*. The set of all graphs may be decomposed into the set with no loops and the set with one loop; the other graphs are of order n^2. The possible configurations of a single loop with a boundary are shown in figure 7. The two and one point functions are then found from the loop gas to be

$$\begin{aligned}
M &\equiv \langle\phi(z_1)\phi(z_2)\rangle = \langle(-1)^{N_{1,1^*}}(-1)^{N_{2,2^*}}\rangle = 1 - 2n(W_{1(2)} + W_{(1)2}) + O(n^2)\\
C_1 &\equiv \langle\phi(z_1)\rangle = \langle(-1)^{N_{1,1^*}}\rangle = 1 - 2n(W_{(1)2} + W_{(12)}) + O(n^2)\\
C_2 &\equiv \langle\phi(z_2)\rangle = \langle(-1)^{N_{2,2^*}}\rangle = 1 - 2n(W_{1(2)} + W_{(12)}) + O(n^2)\,.
\end{aligned}$$

Just as for the case of the loop gas in the bulk, the μ-masses $\{W_i\}$ are the number of weighted loops belonging to the configuration i (see figure 7). They are defined as

$$W_{1(2)} \equiv \sum_{G_{1(2)}} x_c^l\,.$$

Of the four possible configurations, the only one finite in the limit $a \to 0$ is $W_{(12)}$:

$$W_{(12)} = \frac{M - C_1C_2}{4n} + O(n)\,.$$

In terms of η,

$$\begin{aligned}
W_{(12)} = &-\frac{1}{12\pi}\ln(\eta(1-\eta)) - \frac{1}{12\pi}\eta\,{}_3F_2(1,1,\frac{4}{3};2,\frac{5}{3};\eta)\\
&+ \frac{\Gamma(2/3)^2}{6\pi\Gamma(4/3)}(-\eta(1-\eta))^{\frac{1}{3}}\,{}_2F_1(\frac{2}{3},1;\frac{4}{3};\eta) + O(n)\,.
\end{aligned} \tag{42}$$

Note, as for the bulk case, that the term involving σ in M is cancelled by the subtraction of the product of one point functions.

7 Interpretation as a stochastic process

We have seen in section 4.1 that the μ-mass of loops which wind around two of four points is invariant under conformal transformations. In this section, we show that the differential equations they satisfy can be interpreted in terms of an SLE process. Recall that these functions were identified as semi-connected four point functions, for example equation (26)

$$W_{z_1 z_4|z_2 z_3} = \lim_{n\to 0}(8n)^{-1}\Big[\langle\phi(z_1,\overline{z_1})\phi(z_2,\overline{z_2})\phi(z_3,\overline{z_3})\phi(z_4,\overline{z_4})\rangle \\ - C_{12}C_{34} - C_{13}C_{24} + C_{14}C_{23}\Big]\,,$$

where $C_{\{ij\}}$ are the correlation functions of a pair of twist operators at the points $(z_i,\overline{z_i})$ and $(z_j,\overline{z_j})$. The two point function is fixed by scale invariance and the four point function was determined as a solution to the partial differential equations (15) and (16). Using these, we find that $W_{z_1z_4|z_2z_3}$ satisfies the PDE

$$\Big[\frac{3}{2}\partial^2_{z_1} + \sum_{i\neq 1}\frac{\partial_{z_i}}{z_i - z_1}\Big]W_{z_1z_4|z_2z_3} = \frac{1}{24\pi}\Big[\frac{1}{(z_4-z_1)^2} + \frac{1}{z_3-z_1}\Big(\frac{1}{z_3-z_4}+\frac{1}{z_2-z_3}\Big) \\ + \frac{1}{z_2-z_1}\Big(\frac{1}{z_2-z_4}+\frac{1}{z_3-z_2}\Big) + \frac{1}{z_4-z_1}\Big(\frac{1}{z_4-z_3}+\frac{1}{z_4-z_2}\Big)\Big] + O(n)\,, \tag{43}$$

together with a similar equation in which all the z s are replaced by $\bar z$. Note that in the BPZ equations z and $\bar z$ can be taken as independent complex numbers, and it is only in applying these equations to physical quantities that one needs to impose reality conditions. These equations have almost the second order linear form which would result from applying the Itô formula to a martingale of an SLE process started at z_1. One difference is that they are complex. We can obtain a real equation by, for example, taking the real part of the sum of the holomorphic and antiholomorphic equations (the more general case will be discussed below), but note that when we do this we have the sum

$$\partial^2_{z_1} + \partial^2_{\bar z_1} = (\partial_{z_1} + \partial_{\bar z_1})^2 - 2\partial_{z_1}\partial_{\bar z_1} = \partial^2_{x_1} - \frac{1}{2}\Delta_{z_1}\,,$$

where $x_1 = \Re e(z_1)$ and Δ_{z_1} is the Laplacian operator $\partial^2_{x_1} + \partial^2_{y_1}$. We also note that we can remove the inhomogenous part in (43) by defining

$$\widetilde{W} \equiv W - (24\pi)^{-1}2\Re e\big[-2\ln(z_4-z_1)+\ln(z_3-z_4)+\ln(z_2-z_4)-\ln(z_2-z_3)\big] \tag{44}$$

Then we can rewrite the real part of (43) as

$$\Big[3\partial^2_{x_1} + 2\Re e\sum_{i\neq 1}\frac{2\partial_{z_i}}{z_i - z_1}\Big]\widetilde{W} = \frac{3}{2}\Delta_{z_1}\widetilde{W}\,. \tag{45}$$

Consider now a sequence of conformal maps $g_t(z)$ which satisfy the stochastic chordal Loewner equation

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - z_{1t}}\,, \tag{46}$$

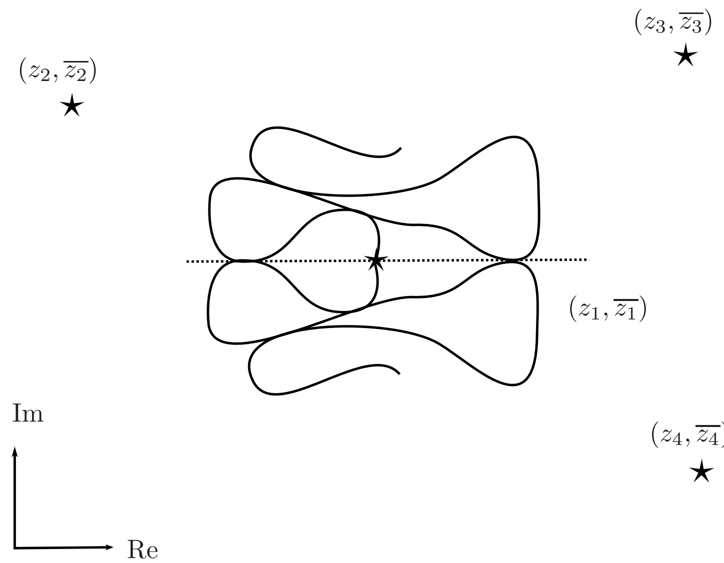

Figure 8: A chordal $\mathrm{SLE}_{8/3}$ growing in the half-plane $\Im m(z) > \Im m(z_1)$ with its reflection in $\Im m(z) = \Im m(z_1)$

where $z_{1t} = z_1 + \sqrt{\kappa} B_t$. Then if $M\big(z_{1t}, g_t(z_j)\big)$ were a martingale, its expectation value would satisfy (45) with the right-hand side set equal to zero. The coefficient of $\partial^2_{x_1}$ would in general be $\kappa/2$, so we should take $\kappa = 6$. The right-hand side of (45), together with the subtractions in (44), therefore express the degree to which the weighted number of loops separating (z_1, z_4) from (z_2, z_3) fails to be a martingale.

Eq. 46 describes a chordal SLE growing into the half-plane $\Im m(z) > \Im m(z_1)$, together with its exact reflection in the line $\Im m(z) = \Im m(z_1)$ as in figure 8 [11]. For $\kappa > 4$ the hull, or outer perimeter, of this curve encloses a growing region of the plane. The entire interior of this hull at time t is mapped into the point $\sqrt{\kappa} B_t$. Because of this, we would not expect the weighted number of loops to be a martingale. Given some initial population of loops, each time that part of a loop is swallowed by the growing hull, it disappears from the population. The right-hand side of (45) must therefore express the rate at which this happens. It is reasonable that this should be proportional to $\Delta_{z_1} W$: if W were constant in some region, this would mean there were no loops passing through it. The leading non-zero rotationally invariant contribution should therefore be proportional to the Laplacian.

Apart from this, the left-hand side expresses the fact that the μ-mass of loops which separate $z_1 z_4$ and $z_2 z_3$ and which have not yet been (partly or wholly) swallowed by the hull is the same as that in the conformally equivalent case of loops which separate $(z_{1t}, g_t(z_4))$ from $(g_t(z_2), g_t(z_3))$. This breaks down when the hull swallows any of the other three points. We conjecture that the subtractions in (44) take this into account.

Of course, this is only suggestive and a number of important issues would have

to be resolved before one could actually derive our results from SLE. In particular, one should explain why it is necessary to consider SLE_6 rather than some other value of κ. This is presumably related to the fact that the hull of SLE_6 corresponds locally to $\mathrm{SLE}_{8/3}$, and that both correspond to CFTs with central charge $c = 0$. The choice of a chordal SLE reflected in $\Im m(z) = \Im m(z_1)$ is clearly arbitrary; other choices correspond to taking different linear combinations of the holomorphic and anti-holomorphic equations. Perhaps using radial or whole-plane SLE would make the formula look more symmetrical.

8 Conclusion

In this paper we have studied scaling properties of loops in the loop gas picture of the O(n) model. In the picture, the partition function is a sum over all graphs of non-intersecting closed loops, weighted by nx_c^l where x is a function of the reduced coupling and l is the length of the loop. We introduced twist operators, whose correlation functions count the loops separating the locations of the operators with weight n' different to the usual weight n, or equivalently count the minimum number of crossings of defect lines running between these locations. For the particular choice $n' = -n$, the twist operators have level two null states and their correlation functions satisfy BPZ type partial differential equations on the Riemann sphere. Thus, conformal field theory may be used to determine the analytic form of the two and four point functions. In the loop gas picture, the choice $n' = -n$ means that loops are counted are weighted by an additional factor of -1 to the power of the number of defect lines crossed, and the choice of path for the defect lines between the locations of the operators is unimportant.

The limit $n \to 0$ describes the theory of self-avoiding loops. In this limit, the partition function and correlation functions are dependent to first order in n only on the configurations of a single loop. Hence, by equating particular semi-connected parts of the four point function calculated using conformal field theory to the result from the Coulomb gas picture, we have deduced the expected number of weighted loops winding around two of the four points and shown that this number is invariant under conformal transformations. This is presumably the mass of such a subset under the measure on loops introduced by Werner. Other configurations of loops receive contributions from vanishingly small loops around a single point and are not finite in the limit of vanishing lattice spacing.

The central charge of the O(n) model for small n may be found from the dependence of the μ-mass of loops around two of the four points as a function of the cross ratio of the positions of the four operators. The result $c \sim 5n/3\pi$ from the analytic results agrees with other methods of calculating the central charge of the O(n) model, and lends support to the interpretation of the stress tensor for curves satisfying conformal restriction given in Ref.[7].

A similar calculation was also carried out for the model defined in a simply connected domain. A general simply connected domain may be mapped via a conformal transformation to the upper half plane with the real axis as the boundary. The two point function of operators in the upper half plane satisfies

the same partial differential equations as the four point function on the Riemann sphere, with two additional operators positioned at the complex conjugates of the original operators. By again equating the results from conformal field theory with those from the Coulomb gas picture, we have deduced the μ-mass of loops around the two points in the upper half plane.

Finally, we have shown that the differential equations satisfied by the above quantities should have a stochastic interpretation in terms of a chordal SLE_6 process starting from one of the points z_1, along with its reflection in a fixed line.

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